Mathematica Moravica
Vol. **17-2** (2013), 29–37

Characterization of Curves in $\mathbb{E}^{2n+1}$ with 1-type Darboux Vector

H. Kocayiğit, G. Öztürk, B. (Kiliç) Bayram, B. Bulca, and K. Arslan

Abstract. In this study, we give some characterizations on the Darboux instantaneous rotation vector field of the curves in Euclidean $(2n+1)$-space $\mathbb{E}^{2n+1}$ by using Laplacian operator. Further, we give necessary and sufficient conditions for unit speed space curves to have 1-type Darboux vector.

1. Introduction

In the local differential geometry, the characterizations of the curves are very important and fascinating problem. Especially, finding a relation to characterize special curves has an important role in the curve theory. The well-known of these special curves in $\mathbb{E}^3$ is constant slope curve or general helix which is defined by the property that the tangent vector of the curve makes a constant angle with a fixed straight line (the axis of the general helix). A classical result stated by M. A. Lancret in 1802 and first proved by B. de Saint Venant in 1845 (see [24] for details) is: A necessary and sufficient condition that a curve be a general helix is that the ratio of curvature to torsion be constant. Further, many mathematicians focused their study on these special curves in different spaces such as Euclidean space ([9, 10, 12, 18, 20, 23]) and Minkowski space ([5, 11, 16, 21]). Moreover in [17] Mağden, gave a similar characterization for the curves in Euclidean 4-space to be constant slope curve.

The notion of a generalized helix in $\mathbb{E}^3$ can be generalized to higher dimensions in many ways. In [6] the same definition is proposed but in $\mathbb{E}^m$. In [8] the definition is more restrictive: the fixed direction makes a constant angle with all vectors of the Frenet frame. It is easy to check that this definition only works in the odd dimension, i.e. case $m = 2n + 1$. Moreover, in the same reference, it is proven that the definition is equivalent to the fact

2010 *Mathematics Subject Classification.* 14H45, 14H50, 53A04, 58E20.
Key words and phrases. Darboux vector, biharmonic curves, Helices.

that the ratios $\frac{\kappa_1}{\kappa_2}, \frac{\kappa_3}{\kappa_4}, \ldots, \frac{\kappa_{2n-1}}{\kappa_{2n}}$, κ_i being curvatures, are constant. This statement is related with the Lancret Theorem for generalized helices in $\mathbb{E}^3$.

Furthermore, Chen and Ishikawa classified biharmonic curves, the curves for which $\Delta H = 0$ in semi-Euclidean space where Δ is the Laplacian operator and H is the mean curvature vector field of a Frenet curve [5]. After them, Kocayiğit studied biharmonic curves and 1-type curves, i.e. the curves for which $\Delta H = \lambda H$ holds, where λ is constant, in Euclidean 3-space $\mathbb{E}^3$ and Minkowski 3-space $\mathbb{E}_1^3$. In [3], Baros and Gray studied curves in Euclidean space with 1-type mean curvature vector. Further in [13], the authors considered the curves in Euclidean space with 1-type mean curvature vector. For more details see also [1]. Recently, in [14] the present authors classified the unit speed curves in $\mathbb{E}^3$ with harmonic and 1-type Darboux vector respectively.

In this study, we give some characterizations on the Darboux instantaneous rotation vector field of the curves in Euclidean $(2n+1)$-space $\mathbb{E}^{2n+1}$ by using Laplacian operator. Further, we give necessary and sufficient conditions for unit speed space curves to have 1-type Darboux vector.

2. BASIC CONCEPTS

Let $\gamma = \gamma(t) : I \to \mathbb{E}^m$ be a regular curve in $\mathbb{E}^m$ (i.e.$\|\gamma'\|$ is nowhere zero), where I is an interval in $\mathbb{R}$. The curve γ is called a *Frenet curve of rank d (or osculating order d)* ($d \in \mathbb{N}_0$, $d \leq m$) if $\gamma'(t)$,$\gamma''(t)$, $\gamma'''(t), \ldots, \gamma^{(d)}(t)$ are linearly independent and $\gamma'(t)$,$\gamma''(t)$,$\gamma'''(t), \ldots, \gamma^{(d+1)}(t)$ are no longer linearly independent for all $t \in I$. In this case, $Im(\gamma)$ lies in a d-dimensional Euclidean subspace of $\mathbb{E}^m$. For each Frenet curve of rank d, there occur an associated orthonormal d-frame $\{V_1, V_2, \ldots, V_d\}$ along γ, the Frenet d-frame and $d-1$ functions $\kappa_1, \kappa_2, \ldots, \kappa_{d-1} : I \to \mathbb{R}$, and the Frenet curvatures, such that

$$\text{(1)} \qquad \begin{bmatrix} V_1' \\ V_2' \\ V_3' \\ \vdots \\ V_d' \end{bmatrix} = v \begin{bmatrix} 0 & \kappa_1 & 0 & \cdots & 0 \\ -\kappa_1 & 0 & \kappa_2 & \cdots & 0 \\ 0 & -\kappa_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \kappa_{d-1} \\ 0 & 0 & \cdots & -\kappa_{d-1} & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_d \end{bmatrix}$$

where v is the speed of the curve.

Infect, to obtain $V_1, V_2, \ldots, V_d$ it is sufficient to apply the Gram-Schmidt orthonormalization process to $\gamma'(t), \gamma''(t), \gamma'''(t), \ldots, \gamma^{(d)}(t)$. Moreover, the functions $\kappa_1, \kappa_2, \ldots, \kappa_{d-1}$ are easily obtained as by product during this calculation. More precisely, $V_1, V_2, \ldots, V_d$ and $\kappa_1, \kappa_2, \ldots, \kappa_{d-1}$ are determined by the following formulas [7]:

$$(2)\qquad \begin{aligned} E_1(t) &= \gamma'(t); & V_1 &= \frac{E_1}{\|E_1(t)\|} \\ E_r(t) &= \gamma^{(r)}(t) - \sum_{i=1}^{r-1} < \gamma^{(r)}(t), \quad E_i(t) > & &\frac{E_i(t)}{\|E_i(t)\|^2}; \\ \kappa_{r-1}(t) &= \frac{\|E_r(t)\|}{\|E_{r-1}(t)\| \, \|E_1(t)\|}, \\ V_r &= \frac{E_r(t)}{\|Er(t)\|}, \end{aligned}$$

where $r \in \{2, 3, \ldots, d\}$.

3. Darboux Vector and Darboux Vertex of a Curve in $\mathbb{E}^{2n+1}$

Let $\gamma : \mathbb{R} \to \mathbb{E}^m$ be a Frenet curve of osculating order d parametrized by arc length. When the Frenet m-frame $V_1, V_2, \ldots, V_m$ of a curve γ in the Euclidean space $\mathbb{E}^m$ is translated to an arbitrary fixed point O and the arc length s is considered as the time, the motion of the m-frame is a rotation about the point O. If m is odd the rotation about the point O has an instantaneous axis of rotation. If m is even there is no such axis of rotation.

Suppose that $m = 2n+1$. Note by $N_0(s) = \gamma'(s), N_1(s), N_2(s), \ldots, N_{d-1}(s)$ the unit vectors of the $(2n+1)$-frame of γ at s. We define the Darboux vector and Darboux vertices of curves in the Euclidean space $\mathbb{E}^{2n+1}$.

Definition 1. Let γ be a unit speed curve of osculating order d $(3 \leq d \leq 2n+1)$ in $\mathbb{R}^{2n+1}, n \geq 1$. Let us denote

$$(3)\qquad \begin{aligned} a_0 &= \kappa_2 \kappa_4 \cdots \kappa_{d-1} \\ a_1 &= \frac{\kappa_1}{\kappa_2} a_0 \\ &\vdots \\ a_j &= \frac{\kappa_{2j-1}}{\kappa_{2j}} a_{j-1}, \qquad 2 \leq j \leq \frac{d-1}{2}, \\ a_{\frac{d-1}{2}} &= \frac{\kappa_{d-2}}{\kappa_{d-1}} a_{\frac{d-3}{2}} = \kappa_1 \kappa_3 \cdots \kappa_{d-2}. \end{aligned}$$

The Darboux vector in $\mathbb{E}^{2n+1}$ is defined by

$$(4)\qquad W(s) = \sum_{j=0}^{\frac{d-1}{2}} a_j N_{2j} = a_0 N_0 + a_1 N_2 + a_2 N_4 + \cdots + a_{\frac{d-1}{2}} N_{d-1},$$

where $\{N_0 = \gamma'(s), N_1, N_2, \ldots, N_{d-1}\}$ is the Frenet frame of γ [25].

Corollary 1. *Let γ be a Frenet curve of osculating order d $(3 \leq d \leq 2n+1)$ in $\mathbb{E}^{2n+1}, n \geq 1$, then*

$$\frac{a_j}{a_{j-1}} = \frac{\kappa_{2j-1}}{\kappa_{2j}} \tag{5}$$

where $1 \leq j \leq \frac{d-1}{2}$.

Lemma 1 ([25])**.** *The derivative of the Darboux vector W(s) is*

$$W'(s) = \sum_{j=0}^{\frac{d-1}{2}} a'_j N_{2j} = a'_0 N_0 + a'_1 N_2 + a'_2 N_4 \cdots + a'_{\frac{d-1}{2}} N_{d-1}. \tag{6}$$

Definition 2. The point $\gamma(s_0)$ is called Darboux vertex of γ if the first derivative of the Darboux vector $W(s)$ is vanishing at that point.

Theorem 1. *Let γ be a Frenet curve of osculating order d $(3 \leq d \leq 2n+1)$ in $\mathbb{E}^{2n+1}, n \geq 1$. Then the curve has a Darboux vertex at point $\gamma(s_0)$ if and only if*

$$\left(\frac{\kappa_1}{\kappa_2}\right)' = 0, \quad \left(\frac{\kappa_3}{\kappa_4}\right)' = 0, \quad \ldots, \quad \left(\frac{\kappa_{d-2}}{\kappa_{d-1}}\right)' = 0 \tag{7}$$

at point $s = s_0$.

Definition 3. The Laplacian operator of the Darboux vector W of γ is defined by

$$\Delta W = -\nabla_{\gamma'(s)} \nabla_{\gamma'(s)} W = -\nabla^2_{\gamma'(s)} W, \tag{8}$$

where ∇ is the Levi-Civita connection given by $\nabla_{\gamma'(s)} = \frac{d}{ds}$.

Lemma 2. *The Laplacian operator of the Darboux vector W of γ is*

$$-\Delta W = W''(s) = \sum_{j=0}^{\frac{d-1}{2}} a''_j N_{2j} + \sum_{i=0}^{\frac{d-3}{2}} (a'_i \kappa_{2i+1} - a'_{i+1} \kappa_{2i+2}) N_{2i+1}. \tag{9}$$

Proof. Differentiating (6) with respect to s we get

$$-\Delta W = W''(s) = \sum_{j=0}^{\frac{d-1}{2}} a''_j N_{2j} + \sum_{j=0}^{\frac{d-1}{2}} a'_j N'_{2j} \tag{10}$$

where $N'_{2j} = \nabla_{\gamma'(s)} N_{2j}$. Further, by the use of the Frenet formulas (1) we get the desired equation (9). □

Definition 4. A regular curve γ in $\mathbb{E}^{2n+1}$ is said to have harmonic Darboux vector if

$$\Delta W = 0 \tag{11}$$

holds. Further, a regular curve γ in $\mathbb{E}^{2n+1}$ is said to have *1-type Darboux vector* if the condition

$$\Delta W = \lambda W, \quad \lambda \in \mathbb{R} \tag{12}$$

holds.

By the use of (9) we get the following result.

Theorem 2. *Let γ be a Frenet curve of osculating order d $(3 \leq d \leq 2n+1)$ in $\mathbb{E}^{2n+1}$, $n \geq 1$. Then γ has harmonic Darboux vector field if and only if*

$$\begin{aligned} \frac{a'_{i+1}}{a'_i} &= \frac{\kappa_{2i+1}}{\kappa_{2i+2}}, \quad 0 \leq i \leq \frac{d-3}{2}, \\ a''_j &= 0, \qquad\quad 0 \leq j \leq \frac{d-1}{2} \end{aligned} \tag{13}$$

holds.

Thus, we have the following corollary of the theorem.

Corollary 2. [14] *Let $\gamma = \gamma(s) : I \to \mathbb{E}^3$ be a unit speed curve in Euclidean* 3*-space $\mathbb{E}^3$. Then γ has harmonic Darboux vector field if and only if*

$$\kappa_1 = c\kappa_2, \quad \kappa_2 = c_1 s + c_2 \tag{14}$$

hold, where $c, c_1, c_2 \in \mathbb{R}$.

By the use of (12) we get the following result.

Theorem 3. *Let γ be a Frenet curve of osculating order d $(3 \leq d \leq 2n+1)$ in $\mathbb{E}^{2n+1}$, $n \geq 1$. Then γ has 1-type Darboux vector field if and only if*

$$\begin{aligned} \frac{a'_{i+1}}{a'_i} &= \frac{\kappa_{2i+1}}{\kappa_{2i+2}}, \quad 0 \leq i \leq \frac{d-3}{2}, \\ a''_j &= \lambda a_j, \qquad 0 \leq j \leq \frac{d-1}{2} \end{aligned} \tag{15}$$

holds.

Thus, we have the following corollary of the theorem.

Corollary 3 ([14]). *Let $\gamma = \gamma(s) : I \to \mathbb{E}^3$ be a unit speed curve in Euclidean 3-space $\mathbb{E}^3$. Then γ is of harmonic 1-type Darboux vector field if and only if*

$$\begin{aligned} &\kappa_1 = \lambda\kappa_2, \\ &\kappa_2 = c_1 \exp(\sqrt{\lambda} s) + c_2 \exp(-\sqrt{\lambda} s), \end{aligned} \tag{16}$$

holds, where $\lambda, c_1, c_2 \in \mathbb{R}$.

4. General Helices

A general helix is a curve $\gamma : \mathbb{R} \to \mathbb{E}^m$ such that its tangent vector forms a constant angle with a given direction v at $\mathbb{E}^m$. It is not difficult to see that this is equivalent to asking that the tangent indicatrix of γ, $\gamma_T : \mathbb{R} \to \mathbb{S}^{m-1} \subset \mathbb{E}^m$ is contained in a $(m-2)$-sphere in $\mathbb{S}^{m-1}$. In particular, we have that this $(m-2)$-sphere is of maximum radius (or an equator) if and only if γ_T is a $(m-1)$ flat curve, in the sense that it lies a hyperplane of $\mathbb{E}^m$ (orthogonal to the direction v). So $(m-1)$ flat curves can be regarded as a particular case of generalized helix in $\mathbb{E}^m$ [6].

Proposition 1 ([6]). *A curve $\gamma : \mathbb{R} \to \mathbb{E}^{2n+1}$ is a generalized helix of osculating order d $(3 \leq d \leq 2n+1)$ in $\mathbb{E}^{2n+1}(n \geq 1)$ if and only if the function $\det(\gamma''(s), \gamma'''(s), \ldots, \gamma^{(d+1)}(s))$ is identically zero, where $\gamma^{(i)}(s)$ represents the $i-th$ derivative of γ with respect to its arc length.*

Remark 1. In [8] the definition of general helix of order d in $\mathbb{E}^{2n+1}$ is more restrictive. It is easy to check that this definition works in the odd dimensional case. Moreover in the same reference it is proven that the definition is equivalent to the fact that the ratios $\frac{\kappa_1}{\kappa_2}, \frac{\kappa_3}{\kappa_4}, \ldots, \frac{\kappa_{d-2}}{\kappa_{d-1}}$ are constant, where $3 \leq d \leq 2n+1$.

Definition 5. A Frenet curve of rank d for which $\kappa_1, \kappa_2, \ldots, \kappa d - 1$ are constant is called generalized screw line or helix [4]. Since these curves are trajectories of the 1-parameter group of the Euclidean transformations, in [15], Klein and Lie called them W-curves.

A unit speed W-curve of osculating order d $(3 \leq d \leq 2n+1)$ in $\mathbb{E}^{2n+1}$ has the parametrization of the form

$$\gamma(s) = a_0 + b_0 s + \sum_{i=1}^{\frac{d-1}{2}} (a_i \cos \mu_i s + b_i \sin \mu_i s) \tag{17}$$

where $a_0, b_0, a_1, \ldots, a_{\frac{d-1}{2}}, b_1, \ldots, b_{\frac{d-1}{2}}$ are constant vectors in $\mathbb{E}^{2n+1}$ and $\mu_1 < \mu_2 < \cdots < \mu_{\frac{d-1}{2}}$ are positive real numbers. So, a W-curve of rank 3 is a right circular helix.

Remark 2. Every W-curve in $\mathbb{E}^{2n+1}$ can be regarded as a general helix in $\mathbb{E}^{2n+1}$. But the converse statement is not true.

Definition 6. Let $\gamma : I \subset \mathbb{R} \to \mathbb{E}^{2n+1}$ be a unit speed curve of osculating order d $(3 \leq d \leq 2n+1)$. The functions $H_j : I \to \mathbb{R}$ defined by

$$\begin{aligned} &H_0 = 0, \quad H_1 = \frac{\kappa_1}{\kappa_2}, \\ &H_j = \{\nabla_{\gamma'(s)} H_{j-1} + H_{j-2}\kappa_j\} \frac{1}{\kappa_{j+1}}, \qquad 2 \leq j \leq d-2 \end{aligned} \tag{18}$$

are called the harmonic curvatures of γ, where $\kappa_1, \kappa_2, \ldots, \kappa_{d-1}$ are Frenet curvatures of γ which are not necessarily constant and ∇ is the Levi-Civita connection [19]. For more details see also [2].

By the use of (18) with (13) we get the following result.

Proposition 2. *Let $\gamma : I \subset \mathbb{R} \to \mathbb{E}^{2n+1}$ be a unit speed curve of osculating order d $(3 \leq d \leq 2n+1)$. If γ has constant harmonic curvatures, then*

$$\begin{aligned} H_{2r} &= 0, & 1 \leq r \leq \frac{d-1}{2} \\ H_{2r-1} &= \frac{\kappa_1}{\kappa_2}\frac{\kappa_3}{\kappa_4}\cdots\frac{\kappa_{2r-1}}{\kappa_{2r}}, & 1 \leq r \leq \frac{d-1}{2}. \end{aligned} \tag{19}$$

We obtain the following result.

Theorem 4. *Let $\gamma : I \subset \mathbb{R} \to \mathbb{E}^{2n+1}$ be a unit speed curve of osculating order d $(3 \leq d \leq 2n+1)$. If the Darboux vector $W(s)$ of γ is harmonic then the harmonic curvatures are constant functions of the form*

$$H_{2r-1} = \frac{a_r}{a_0}, \quad H_{2r} = 0, \quad 1 \leq r \leq \frac{d-1}{2} \tag{20}$$

where

$$a_i = c_i s + d_i, \quad i = 0, \ldots, \frac{d-1}{2}. \tag{21}$$

Proof. Using (5) with (19) we get the result. □

REFERENCES

[1] K. Arslan, Y. Aydın, G. Öztürk and H. Ugail, *Biminimal Curves in Euclidean Spaces*, International Electronic Journal of Geometry, **2** (2009), 46-52.

[2] K. Arslan, Y. Celik and H. H. Hacısalihoğlu, *On harmonic curvatures of a Frenet curve*, Common. Fac. Sci. Univ. Ank. Series A1, **49** (2000), 15-23.

[3] M. Barros and O. J. Gray, *On Submanifolds with Harmonic Mean Curvature*, Proc. Amer. Math. Soc., **123** (1995), 2545-2549.

[4] B. Y. Chen, J. Deprez and P. Verheyen, *Immersions with geodesics of 2-type*, In Geometry and Topology of Submanifolds, IV, Proceedings of the Conference on Differential Geometry and Vision, Leuven 27–29 June 1991 (Dillen, F., ed.). World Scientific, Singapore, (1992), 87-110.

[5] B. Y. Chen and S. Ishikawa, *Biharmonic surface in pseudo-Euclidean spaces*, Mem. Fac. Sci. Kyushu Univ. A, **45** (1991), 323-347.

[6] M.C. Romero-Fuster, E. Sanabria-Codesal, *Generalized helices, twistings and flattenings of curves in n-space*, Matematica Contemporanea, **17** (1999), 267-280.

[7] H. Gluck, *Higher curvatures of curves in Euclidean space*, Am. Math. Monthly, **73** (1966), 699-704.

[8] [4] H. A. Hayden, *On a generalized helix in a Riemannian n-space*, Proc. London Math. Soc., **32** (1931), 37-45.

[9] H. H. Hacısalihoğlu and R. Öztürk, *On the characterization of inclined curves in* $\mathbb{E}^n$ *I*, Tensor, New Series, **64** (2003), 157-162.

[10] H. H. Hacısalihoğlu and R. Öztürk, *On the characterization of inclined curves in* $\mathbb{E}^n$ *II*, Tensor, New Series, **64** (2003), 163-169.

[11] K. Ilarslan, E. Nesovic and M. Petrovic-Torgasev, *Some characterizations of rectifying curves in the Minkowski 3-Space*, Novi Sad J. Math., **33** (2003), 23-32.

[12] S. Izumiya, N. Takeuchi, *New special curves and developable surfaces*, Turk J. Math., **28** (2004), 153-163.

[13] B. Kılıç, K. Arslan, *Harmonic 1-type submanifolds of Euclidean spaces*, Int. J. Math. Stat., **3** (2008), 47-53.

[14] H. Kocayiğit, M. Önder and K. Arslan, *Characterization of Space Curves with 1-type Darboux Vector*, Preprint.

[15] F. Klein and S. Lie, *über diejenigen ebenenen kurven welche durch ein geschlossenes system von einfach unendlich vielen vartauschbaren linearen Transformationen in sich übergehen*, Math. Ann., **4** (1871), 50-84.

[16] H. Kocayiğit, M. Önder, *Timelike curves of constant slope in Minkowski space*, BU/JST, **1**(2) (2007), 311-318.

[17] A. Mağden, *On the curves of constant slope*, YYÜ Fen Bilimleri Dergisi, **4** (1993), 103-109.

[18] J. Monterde, *Curves With Constant Curvature Ratios*, Bulletin of Mexican Mathematic Society, 3a serie, **13** (2007), 177-186.

[19] E. Özdamar and H. H.Hacısalihoğlu, *A characterization of inclined curves in Euclidean n-space*, Comm. Fac. Sci-Univ. Ankara, Ser. Al. Math., **24** (1975), 15-23.

[20] G. Özturk, K. Arslan and H. H. Hacisalihoglu, *A characterization of ccr-curves in* R^m, Proceedings of the Estonian Academy of Sciences, **57** (2008), 217-224.

[21] M. Petrovic-Torgasev, E. Sucurovic, *W-curves in Minkowski space-time*, Novi Sad J. Math., **32**(2) (2002), 55-65.

[22] M.C. Romero-Fuster, E. Sanabria-Codesal, *Generalized helices, twistings and flattenings of curves in n-space*, Mat. Cont., **17** (1999), 267-280.

[23] P. D. Scofield, *Curves of constant precession*, Am. Math. Monthly, **102** (1995), 531-537.

[24] D. J. Struik, *Lectures on Classical Differential Geometry*, 2nd ed., Addison Wesley, Dover, 1988.

[25] R. Uribe-Vargas,*On Singularities, 'Perestroikas' and Differential Geometry of Space Curves*, L'Enseigement Mathématique, **50** (2004), 69-101.

H. Kocayiğit
Department of Mathematics
Celal Bayar University
Manisa
Turkey
E-mail address: huseyin.kocayigit@bayar.edu.tr

G. Öztürk
Department of Mathematics
Kocaeli University
Kocaeli
Turkey
E-mail address: ogunay@kocaeli.edu.tr

B. (Kiliç) Bayram
Department of Mathematics
Balikesir University
Balikesir
Turkey
E-mail address: benguk@balıkesir.edu.tr

B. Bulca
Department of Mathematics
Uludag University
16059 Bursa
Turkey
E-mail address: bbulca@uludag.edu.tr

K. Arslan
Department of Mathematics
Uludag University
16059 Bursa
Turkey
E-mail address: arslan@uludag.edu.tr